

Meeting between mathematicians and physicists, IHP, December 2011

**Two component condensates: vortex,
skyrmions and phase separation**

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ANR-blanc-0238, VoLQuan

Joint works with

Peter Mason (PRA 2011)

Peter Mason and Juncheng Wei
(arxiv 1111.6714)

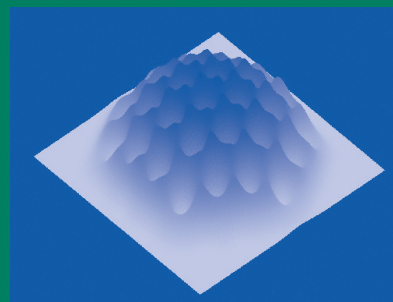
older works with: Xavier Blanc,
Jean Dalibard, Bob Jerrard,
Bernard Helffer, Nicolas Lerner,
Francis Nier, Jimena Royo-
Letelier

Motivated by experiments in the group
of Jean Dalibard (Laboratoire
Kastler Brossel, ENS, Paris).

Progress in Nonlinear Differential Equations
and Their Applications

Amandine Aftalion

Vortices in Bose–Einstein Condensates



Birkhäuser

Outline of the talk

I. Main mathematical results for a single condensate

II. Two component condensates: numerical simulations

III. Two component condensates: rigorous results
(nonlinear σ model)

Gross Pitaevskii energy for a single condensate

A single BEC is in a state which minimizes

$$E(\psi) = \int \frac{1}{2} |\nabla\psi - i\Omega \times r\psi|^2 - \frac{1}{2} \Omega^2 r^2 |\psi|^2 + V(r) |\psi|^2 + \frac{1}{2} g |\psi|^4,$$

Which potential V ? $V(x, y) = x^2 + \alpha^2 y^2$, $V(r) = r^2 + Ae^{-r^2}$

Two mathematical limits

- g large, Thomas Fermi limit: the kinetic energy term is small. (analogue of Bethuel Brezis Helein analysis of vortices, also Jerrard, Sandier-Serfaty). vortex core of size $1/\sqrt{g}$ is much smaller than the distance between vortices. Ignat-Millot, Yngvason et al, Aftalion-Jerrard, Aftalion-Jerrard-Royo Letelier
- Ω small, the ground state is real valued. no vortices in the small density region up to the first critical velocity.
- Rapid rotation: $\Omega \rightarrow 1$. vortex cores start to overlap: reduction to a single particle state: the lowest Landau level (LLL). Aftalion-Blanc-Dalibard, Aftalion-Blanc-Nier

Minimize the Gross Pitaevskii energy in the Thomas Fermi limit, g large:

$$E(\psi) = \int \frac{1}{2} |\nabla \psi - i\Omega \times r \psi|^2 + \frac{1}{2} r^2 |\psi|^2 (1 - \Omega^2) + \frac{1}{2} g |\psi|^4,$$

under $\int_{\mathbb{R}^2} |\psi|^2 = 1$, $r = (x, y)$.

Leading order, the kinetic energy can be neglected, inverted parabola profile:
 $(1 - \Omega^2)r^2 + g|\psi|^2 = \lambda$.

Next order: computation of the critical velocity Ω_c for the nucleation of the first vortex. The ground state stays real valued until Ω_c .

Difficulty, the problem is set on \mathbb{R}^2 with a constraint.

Next order, vortex balls of size $1/\sqrt{g}$. Vortex location minimize the vortex interaction energy

$$\sum_i |p_i|^2 - \sum_{i,j} \log |p_i - p_j|$$

Numerically, almost a triangular lattice.

Minimize the Gross Pitaevskii energy in the fast rotating regime: Ω tends to 1.

$$E(\psi) = \int \frac{1}{2} |\nabla \psi - i\Omega \times r \psi|^2 + \frac{1}{2} r^2 |\psi|^2 (1 - \Omega^2) + \frac{1}{2} g |\psi|^4,$$

under $\int_{\mathbb{R}^2} |\psi|^2 = 1$, $r = (x, y)$.

Project ψ onto the LLL, which is the first eigenspace of $-(\nabla - i\Omega \times r)^2$,
 $LLL = \left\{ v(z) = f(z) e^{-\Omega|z|^2/2}, \quad f \text{ holomorphic} \right\}$ with $z = x + iy$.

Aftalion, Blanc, Dalibard- Aftalion, Blanc, Nier

Explicit expression of an approximate solution of the equation thanks to the expression of the projector onto the LLL:

$$[\Pi f](z) = \frac{1}{\pi} \int_{\mathbb{C}} e^{z\bar{z}'} e^{-|z'|^2} f(z') L(dz')$$

What is a hexagonal lattice? $\mathbb{Z} \oplus \tau\mathbb{Z}$ with $\tau = e^{2i\pi/3}$

Use Jacobi Theta functions to perform a calculation similar to that of Abrikosov and minimize the lattice energy:

$$\inf \left\{ \gamma(\tau) = \frac{\int |u|^4}{(\int |u|^2)^2}, \quad u(z) = f(z)e^{-|z|^2/2h}, \quad |u| \text{ periodic on a lattice } \mathbb{Z} \oplus \tau\mathbb{Z} \right\}$$

Minimum = hexagonal lattice, $\tau = j = e^{\frac{2i\pi}{3}}$.

The minimizer is $u_j(z) = g(z)e^{-|z|^2/2}$ with
 $g(z) = e^{\frac{z^2}{2}} \Theta \left(\sqrt{\frac{\sqrt{3}}{2\pi}} z, \frac{1}{2} + i\frac{\sqrt{3}}{2} \right)$.

$|u_j|$ is a periodic function on a triangular lattice and vanishes on each cell.

The square lattice is a saddle point.

Other potentials: asymmetric potential (Aftalion, Blanc, Lerner): fast rotation, (analysis in the LLL, motivated by a paper of Fetter)

$$\int \frac{1}{2} |\nabla \psi|^2 - \Omega \times r(i\psi, \nabla \psi) + \frac{1}{2} \left((1 - \nu^2)x^2 + (1 + \nu^2)y^2 \right) |\psi|^2 + \frac{1}{2} g |\psi|^4.$$

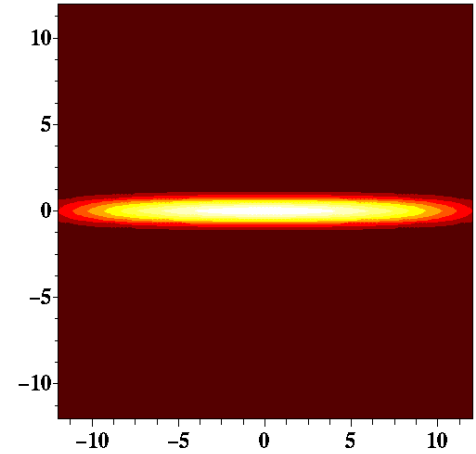
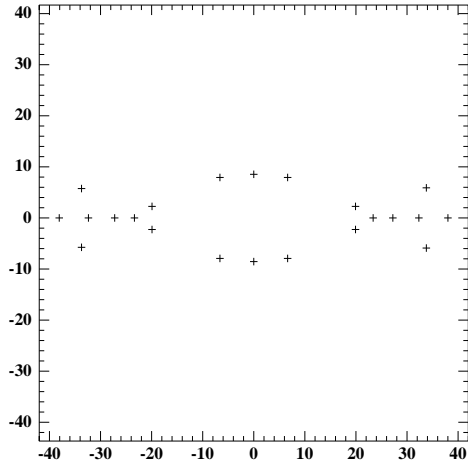
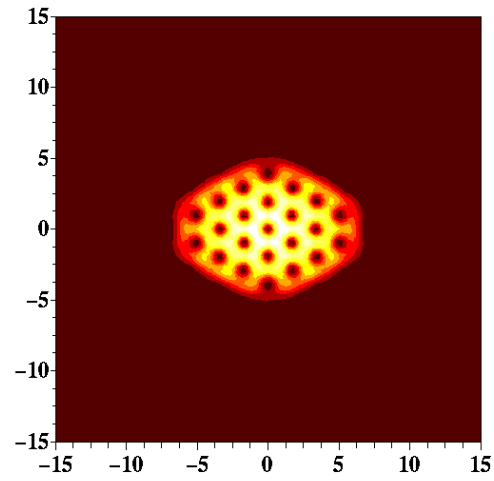
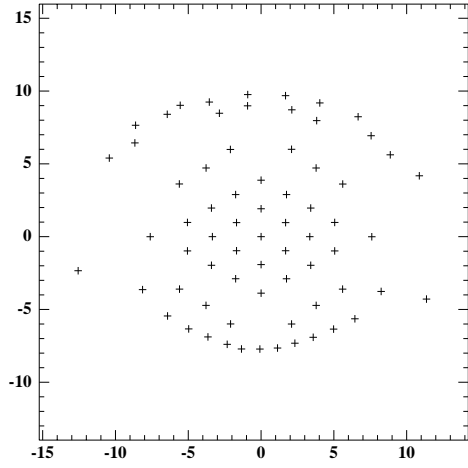
small parameter ε , where $\varepsilon^2 = 1 - \nu^2 - \Omega^2$.

- There are new coordinates (equivalent of the LLL) where the energy gets simplified (diagonalization in a symplectic basis). New energy

$$E_{LLL} = \int \frac{1}{2} (\varepsilon^2 x^2 + \kappa^2 y^2) |\psi|^2 + \frac{1}{2} g |\psi|^4$$

with $\kappa^2 \sim \frac{(\nu^2 + \varepsilon^2/2)(2 - \nu^2)}{(1 - \nu^2)}$. ψ times a gaussian has to be holomorphic.

As ε tends to 0, the condensate expands in the x direction. In the y direction, according to the values of ε and ν , either the condensate expands (inverted parabola profile) or stays bounded.



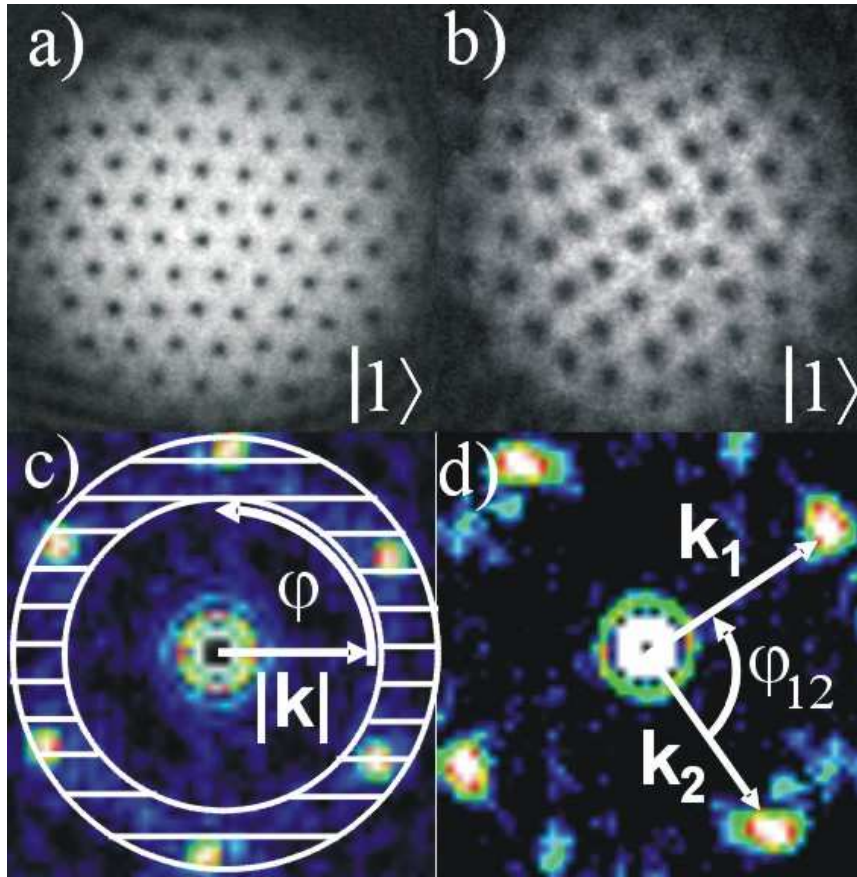
II. Two component condensates: numerical simulations

Kasamatsu, Tsubota, Ueda, Int. J. Mod. Phys. 2005

Cooper, 2008

2 component condensate: 2 wave functions, new phases and defects.

V. Schweikhard, I. Coddington, P. Engels, S. Tung, and E. A. Cornell (2004): a square lattice is stabilized in a two component condensate.



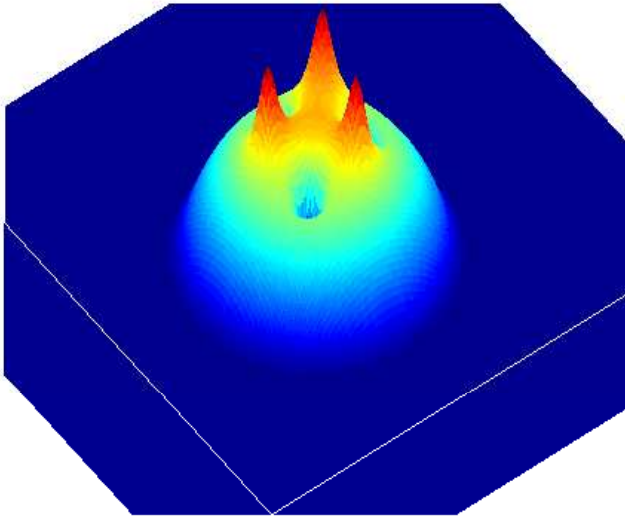
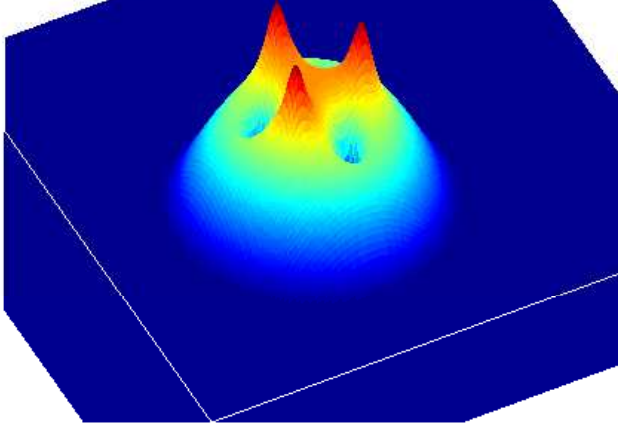
Two component condensates (Aftalion-Mason)

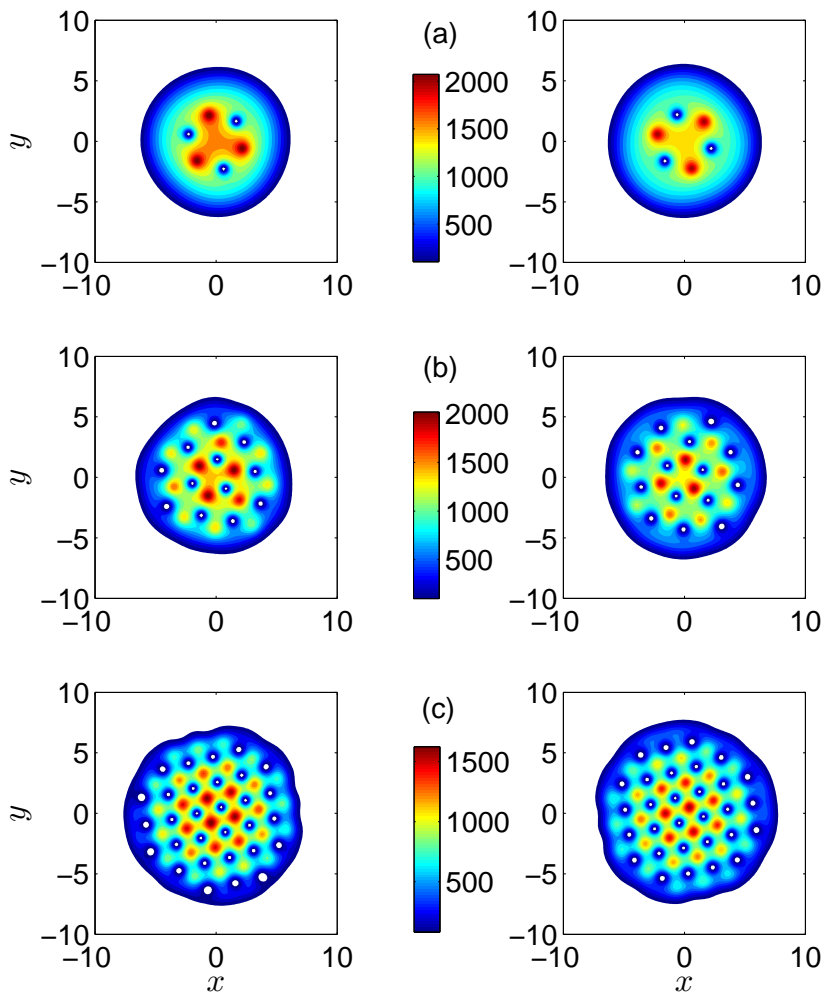
2 different isotopes of the same alkali atom, isotopes of different atoms, or a single isotope in 2 different hyperfine spin states: 2 wave functions ψ_1 and ψ_2 with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$

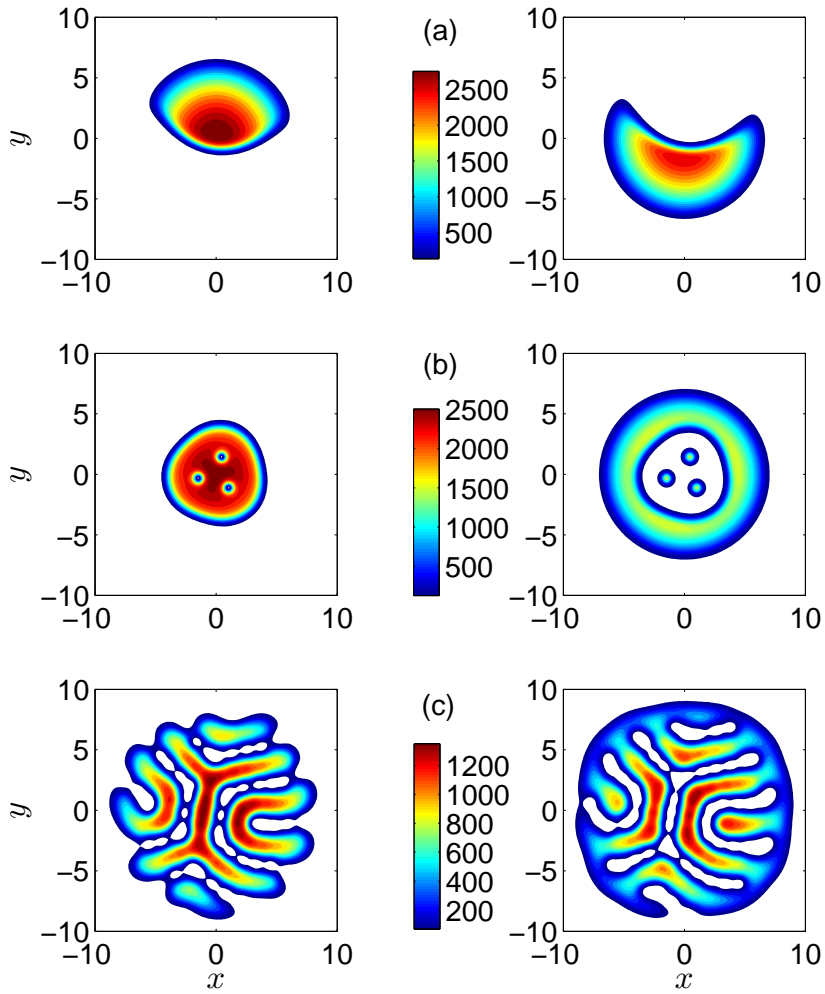
$$E_{\Omega,g}(\psi) = \int \frac{1}{2} |\nabla\psi - i\Omega \times r\psi|^2 + \frac{1}{2} r^2 |\psi|^2 (1 - \Omega^2) + \frac{1}{2} g |\psi|^4,$$

$$E = E_{\Omega,g_1}(\psi_1) + E_{\Omega,g_2}(\psi_2) + g_{12} \int |\psi_1|^2 |\psi_2|^2$$

- g_{12} small: 2 components are disk-shaped with vortex lattices. a vortex in component 1 corresponds to a peak in component 2. Square lattice.
- g_{12} large: phase separation and breaking of symmetry: rotating droplets
- intermediate regime: phase separation but no breaking of symmetry, one component is a disk, the other is an annulus. Skyrmion in the boundary layer
- vortex sheets







g_{12} large:
phase separation

left column
 $|\psi_1|^2$

right column
 $|\psi_2|^2$

$$g_1 = 0.0078,$$

$$g_2 = 0.0083,$$

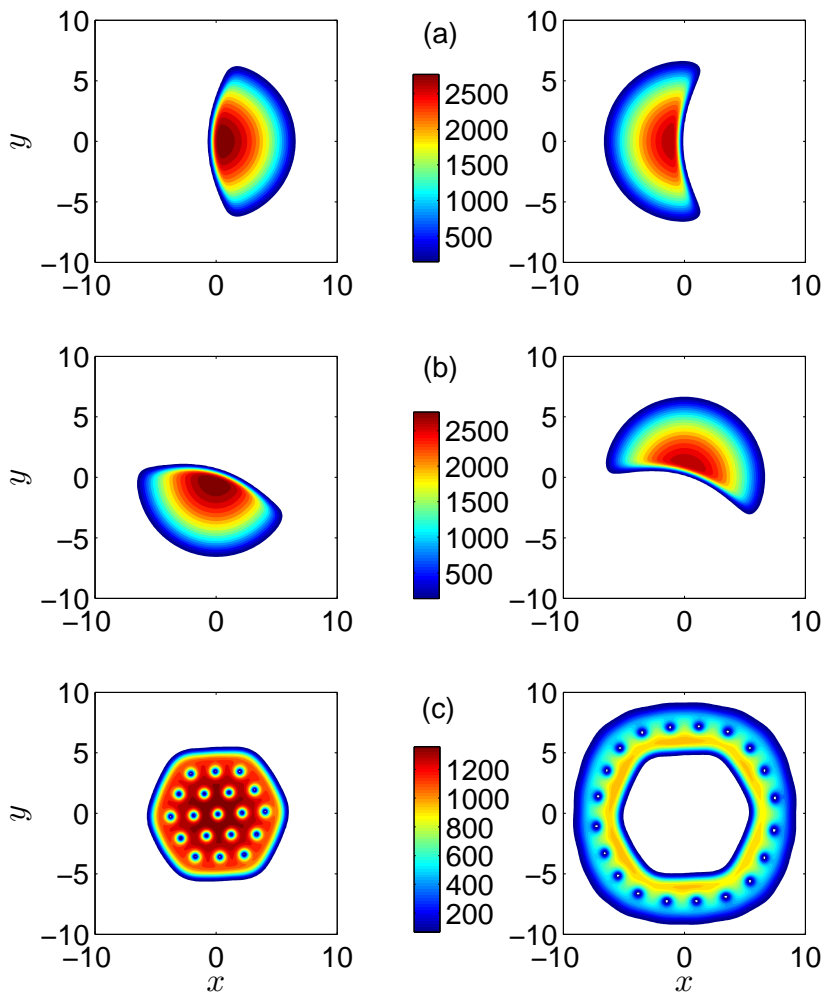
$$N_1 =$$

$$N_2 = 10^5,$$

$$g_{12} = 0.0092,$$

$$\Omega = \text{(a) } 0.1,$$

$$\text{(b) } 0.5, \text{ (c) } 0.9$$



g_{12} larger

left column

$|\psi_1|^2$

right column

$|\psi_2|^2$

$g_1 = 0.0078,$

$g_2 = 0.0083,$

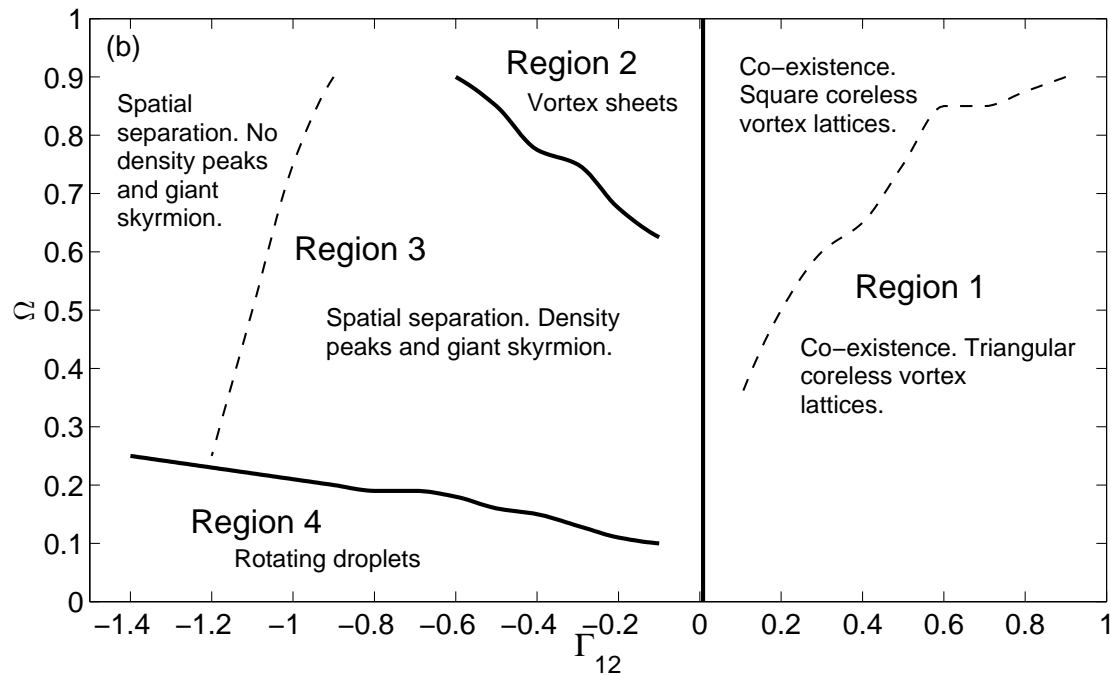
$N_1 = N_2 =$

$10^5, g_{12} =$

$0.0122), \Omega =$

(a) 0, (b) 0.1,

and (c) 0.9.



$\Omega - \Gamma_{12}$ phase
 diagrams $g_1 =$
 0.0078 , $g_2 =$
 0.0083 , $N_1 =$
 $N_2 = 10^5$
 $\Gamma_{12} = 1 - \frac{g_{12}^2}{g_1 g_2}$

III. Two component condensates: rigorous results (nonlinear σ model)

Nonlinear σ model

We define $\rho_T = |\psi_1|^2 + |\psi_2|^2$, $\psi_k = \sqrt{\rho_T} \chi_k$, $\chi_k = |\chi_k| e^{i\theta_k}$ so that $|\chi_1|^2 + |\chi_2|^2 = 1$ and $S_z = |\chi_1|^2 - |\chi_2|^2$. We have $S_z = 1$ when only component 1 is present, $S_z = -1$, when only component 2 is present, and $-1 \leq S_z \leq +1$. The wave functions ψ_1 and ψ_2 respectively represent the up and down components of the spin-1/2 spinor.

The energy can be rewritten in terms of ρ_T , S_z , θ_1 , θ_2 :

$$E = \int \frac{1}{2} (\nabla \sqrt{\rho_T})^2 + \frac{\rho_T (\nabla S_z)^2}{8 (1 - S_z^2)} + \frac{\rho_T}{4} ((1 + S_z) (\nabla \theta_1 - \boldsymbol{\Omega} \times \mathbf{r})^2 + (1 - S_z) (\nabla \theta_2 - \boldsymbol{\Omega} \times \mathbf{r})^2) + \frac{1}{2} (1 - \Omega^2) r^2 \rho_T + \frac{\rho_T^2}{2} (c_0 + c_1 S_z + c_2 S_z^2) d^2 r$$

where $c_0 = \frac{1}{4}(g_1 + g_2 + 2g_{12})$, $c_1 = \frac{1}{2}(g_1 - g_2)$, $c_2 = \frac{1}{4}(g_1 + g_2 - 2g_{12})$,
 $\int \rho_T = N_1 + N_2$, $\int \rho_T S_z = N_1 - N_2$.

More general formula for distinct masses and trapping potentials.

We are left with an energy depending on ρ_T , S_z , θ_1 , θ_2 . In the absence of defects (for instance $\Omega = 0$), it only depends on ρ_T and S_z .

In the Thomas Fermi limit ($g_1 = \alpha_1 g$, $g_2 = \alpha_2 g$, $g_{12} = \alpha_{12} g$ with g large), the kinetic energy term can be neglected and we have to minimize

$$E_{TF} = \int \frac{1}{2}(1 - \Omega^2)r^2 \rho_T + \frac{\rho_T^2}{2} (c_0 + c_1 S_z + c_2 S_z^2)$$

under $\int \rho_T = N_1 + N_2$, $\int \rho_T S_z = N_1 - N_2$.

The reduction of this quadratic form in ρ_T and $\rho_T S_z$ yields ($\Gamma_{12} = 1 - \frac{g_{12}^2}{g_1 g_2}$)

$$E_{TF} = \int \frac{c_2}{2} \left(\rho_T S_z + \frac{c_1}{2c_2} \rho_T - \frac{\lambda}{c_2} \right)^2 + \frac{g_1 g_2 \Gamma_{12}}{8c_2} [\rho_T$$

$$- \frac{c_2}{2g_1 g_2 \Gamma_{12}} (\mu - 2(1 - \Omega^2)r^2) + \frac{c_1}{2g_1 g_2 \Gamma_{12}} (\lambda - 2(1 - \Omega^2)r^2)]^2 + cst$$

The existence of a minimum for this quadratic form implies that $c_2 \geq 0$ and $g_1 g_2 \Gamma_{12} \geq 0$). This implies in particular that $\Gamma_{12} > 0$. **Condition of coexistence**

What can be proved

- If $\Gamma_{12} > 0$, phase coexistence. If g_1, g_2, g_{12} are large, TF limit (Aftalion, Mason, Wei, in progress.)

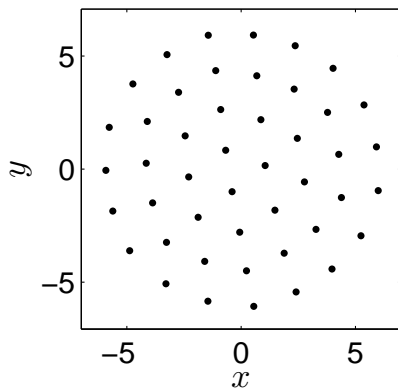
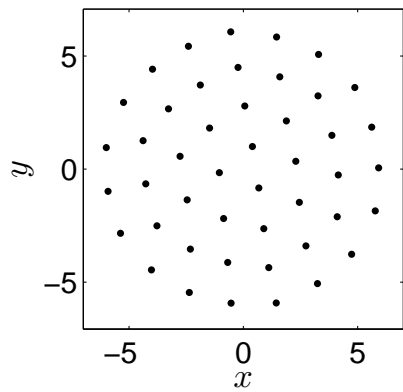
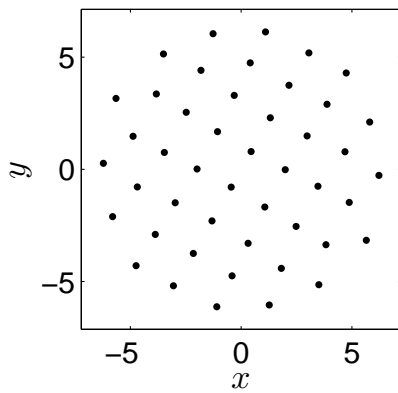
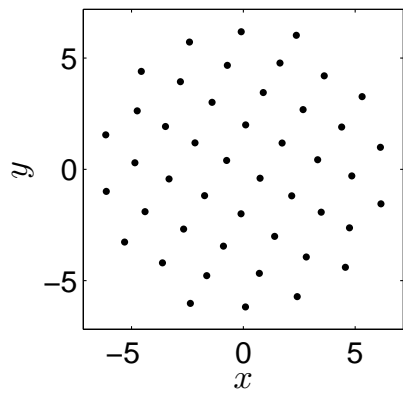
- leading order, inverted parabola profile with 2 disks or a disk and an annulus

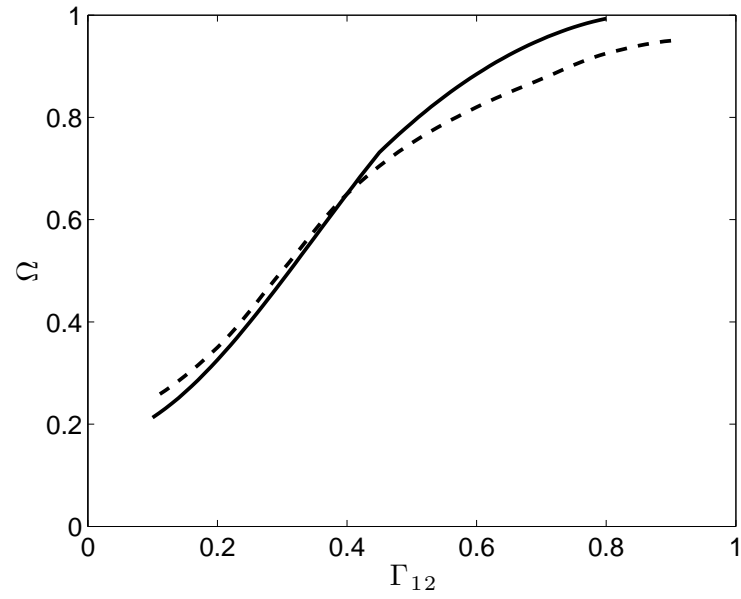
- computation of the critical velocity for the 1st vortex, called Ω_c (in component with larger radius).

- vortex peak interaction (integral similar to that of Eto, Kasamatsu, Nitta, Takeuchi, Tsubota)

$$- \sum_{i,j} (\log |p_i - p_j| + \log |q_i - q_j|) + \sum_i (|p_i|^2 + |q_i|^2) + \sum_{i,j} \frac{c_\Omega}{|p_i - q_j|^2}$$

where p_i are the vortices for component 1, q_j are the vortices for component 2 and $c_\Omega = \frac{\pi(1-\Gamma_{12})|\log g_1|^2}{8\Gamma_{12}^2 N_1 g_1} (2\frac{\Omega}{\Omega_c} - 1)$. At some critical value of c_Ω , the lattice goes from triangular to square: relation between Γ_{12} and Ω .





- $\Gamma_{12} > 0$, $\Omega \rightarrow 1$: LLL regime. The lattice goes from square to triangular (Mueller and Ho, Kasamatsu, Tsubota, Ueda), physicists' results to be rigorously analyzed?
- Can we establish a link between the Ising model $\sum S_{zi}S_{zj}$ and our energy?

$$E_{TF} = \int \frac{1}{2}(1 - \Omega^2)r^2 \rho_T + \frac{\rho_T^2}{2} (c_0 + c_2 S_z^2)$$

Indeed, S_z has a slow varying profile linked to ρ_T , but also varies locally close to a vortex core.

If $\Gamma_{12} < 0$, phase separation is expected.

- **Asymptotic limit $\Gamma_{12} \rightarrow -\infty$** : the coexistence region gets asymptotically small. Proof of the breaking of symmetry for $\Omega = 0$ and g_1, g_2 small (**Jimena Royo-Letelier**) (droplets)

$\Gamma_{12} < 0$ and $c_2 < 0$, then $\Gamma_{12}/c_2 > 0$. Hope to get a TF profile for ρ_{TF} . This requires to understand the equation for S_z at leading order both for droplets and vortex sheets. Link with de Giorgi problems?

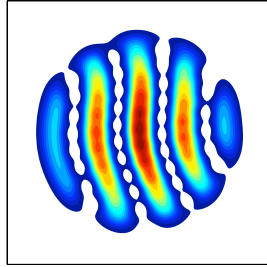
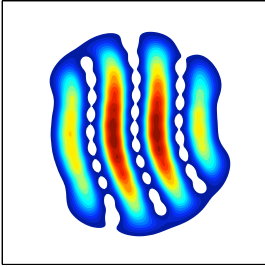
Write $S_z = \cos \phi$, then the energy becomes (for $g_1 = g_2$ and $\Omega = 0$)

$$\int |\nabla \sqrt{\rho_T}|^2 + \frac{\rho_T}{2} |\nabla \phi|^2 + \frac{1}{2} r^2 \rho_T + g_{12} \frac{\rho_T^2}{4} (1 - \cos^2 \phi) + g_1 \frac{\rho_T^2}{4} (1 + \cos^2 \phi)$$

If g_{12} is large, then $\cos^2 \phi = 1$ almost everywhere, except on a boundary layer.

ρ_T is almost TF, and vanishes at interface.

Vortex sheets



Conclusion

Numerical simulations for 2 component condensates. Phase diagrams in Ω and Γ_{12} .

Mathematical contributions: energy estimates to classify the ground states. Characterization of the vortex lattice through a point energy. (LLL regime?)

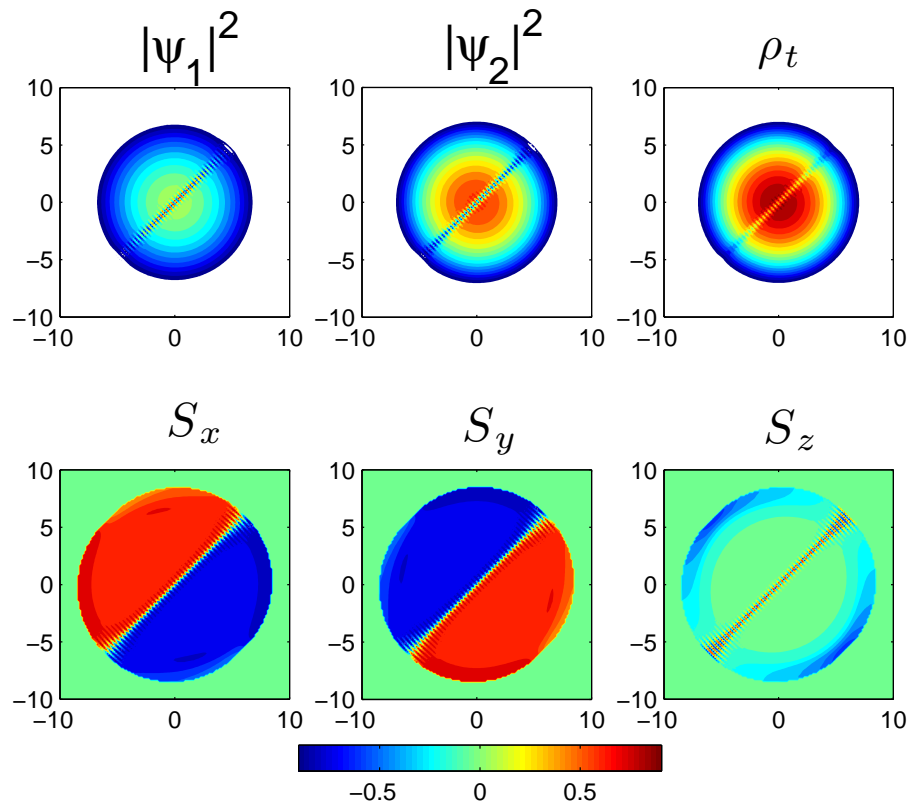
Use of the nonlinear sigma model

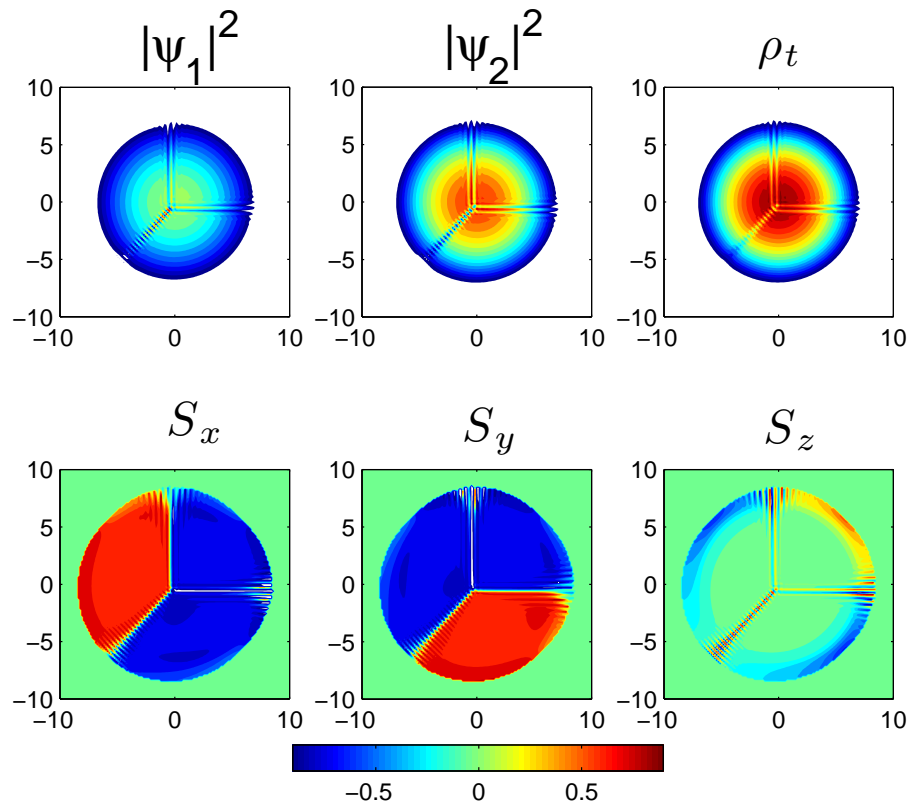
Perspectives: deeper use of the nonlinear sigma model, in this case and for spin orbit coupled condensates.

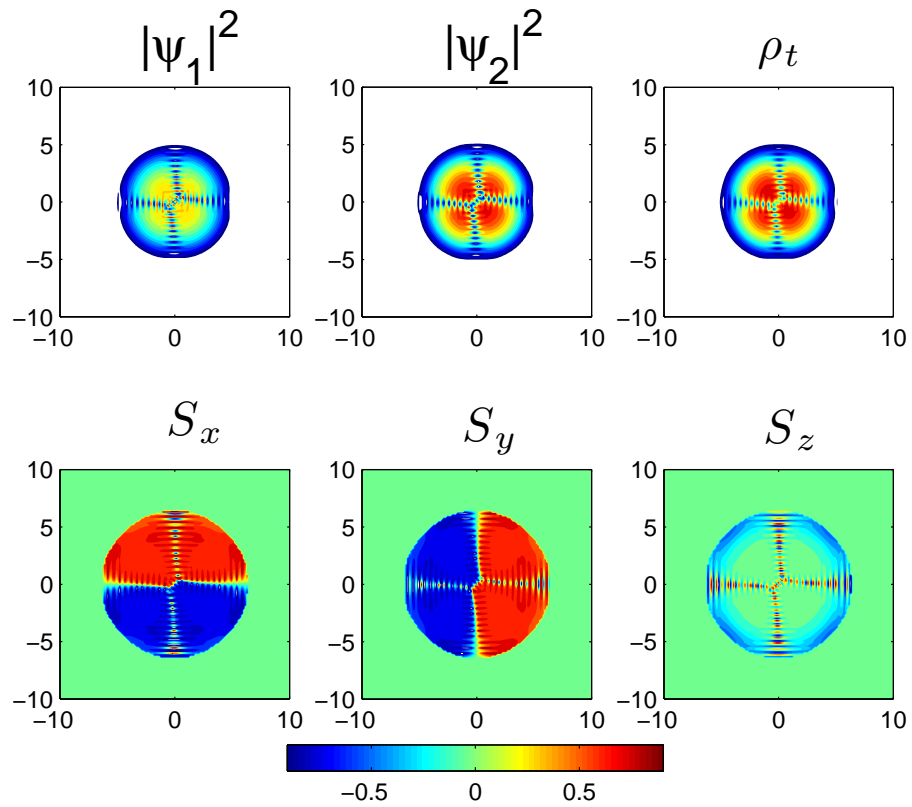
$$\int \sum_{k=1,2} \left(\frac{1}{2} |\nabla \psi_k|^2 + \frac{1}{2} r^2 |\psi_k|^2 - \Omega \psi_k^* L_z \Psi_k + \frac{g_k}{2} |\psi_k|^4 \right) + g_{12} |\psi_1|^2 |\psi_2|^2$$

$$- \kappa \psi_1^* \left(i \frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_2}{\partial y} \right) - \kappa \psi_2^* \left(i \frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_1}{\partial y} \right)$$

$g_1 = 1, g_2 = 2, g_{12} = 0.5, N_1 = N_2 = 1000, \kappa = 10, \Omega = 0.02, 0.05, 0.2$







$$g_1 = g_2 = 4, g_{12} = 2, N_1 = N_2 = 100, \kappa = 0.1, \Omega = 0.9$$

