9th European Conference on Elliptic and Parabolic Problems, Gaeta, May 2016

Two component Bose Einstein condensates: coexistence, segregation and vortex patterns

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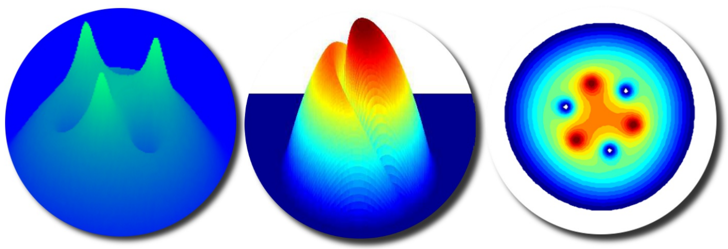
Joint works with

C.Sourdis, to appear in CCM
B. Noris and C. Sourdis (Com. Math. Phys. 2015)
J. Royo-Letelier (Calculus of Variations and PDE's 2015)
P. Mason and J. Wei (Phys. Rev. A 2012)



Motivated from numerical simulations in

Aftalion-Mason (PRA 2012 and PRA 2013).

A two component Bose Einstein condensate is a mixture of 2 species describing:

2 different isotopes of the same alkali atom,
or isotopes of different atoms,
or a single isotope in 2 different hyperfine spin states.

Described by 2 wave functions ψ_1 and ψ_2 with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$ minimizing a Gross Pitaevskii energy with a coupling term.

The coupling can be either through the modulus or through the phase (spin orbit coupling or Rabi coupling).

Gross Pitaevskii energy for a single condensate

A single Bose Einstein condensate is in a state which minimizes

$$E(\psi) = \int_{\mathrm{R}^2} \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2\varepsilon^2} r^2 |\psi|^2 + \frac{g}{2\varepsilon^2}|\psi|^4,$$

under $\int |\psi|^2 = 1$. Mathematical limit: $\varepsilon \to 0$.

$$-\frac{\varepsilon^2}{2}\Delta\psi + \frac{1}{2} r^2 \psi + |\psi|^2\psi = \lambda\psi$$

Leading order, inverted parabola profile:
$|\psi|^2 = \lambda^2 - (1/2)r^2$. Exponential decay at infinity.

Ignat-Millot Uniqueness and convergence to Thomas Fermi profile

Karali-Sourdis very precise estimates. Painlevé boundary layer. See also Gallo-Pelinovski. Based on perturbation arguments to construct an approximate solution, and then use the properties of the linearized operator to get a true solution. The uniqueness implies that it is the ground state.

Two component condensates

2 wave functions ψ_1 and ψ_2 with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$

$$E_g(\psi) = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2\varepsilon^2}r^2|\psi|^2 + \frac{g}{2\varepsilon^2}|\psi|^4,$$

$$E = E_{g_1}(\psi_1) + E_{g_2}(\psi_2) + \frac{g_{12}}{\varepsilon^2}\int |\psi_1|^2|\psi_2|^2$$

ε: small parameter

Important parameter $\Gamma_{12} = 1 - \frac{g_{12}^2}{g_1 g_2}$.

$\Gamma_{12} > 0$: coexistence of the components

$\Gamma_{12} < 0$: segregation (breaking of symmetry)

We look for positive solutions of

$$-\frac{1}{2}\Delta\psi_1 + \frac{\psi_1}{\varepsilon^2}(g_1|\psi_1|^2 + g_{12}|\psi_2|^2) = \frac{1}{\varepsilon^2}\psi_1(\lambda_1 - \frac{1}{2}r^2)$$

$$-\frac{1}{2}\Delta\psi_2 + \frac{\psi_2}{\varepsilon^2}(g_{12}|\psi_1|^2 + g_2|\psi_2|^2) = \frac{1}{\varepsilon^2}\psi_2(\lambda_2 - \frac{1}{2}r^2)$$

with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$. Thomas-Fermi profile given by

$$g_1|\psi_1|^2 + g_{12}|\psi_2|^2 = \lambda_1 - \frac{1}{2}(1-\Omega^2)r^2$$

$$g_{12}|\psi_1|^2 + g_2|\psi_2|^2 = \lambda_2 - \frac{1}{2}(1-\Omega^2)r^2$$

In the case $\Gamma_{12} > 0$, there is a solution to the reduced limiting system, there is a unique positive solution and we can analyze the convergence.

In the case $\Gamma_{12} < 0$, there is a Γ convergence to an interface problem.

Coexistence case, $\Gamma_{12} > 0$

leading order, inverted parabola profile:

$$g_1|\psi_1|^2 + g_{12}|\psi_2|^2 = \lambda_1 - \frac{1}{2}(1-\Omega^2)r^2$$

$$g_{12}|\psi_1|^2 + g_2|\psi_2|^2 = \lambda_2 - \frac{1}{2}(1-\Omega^2)r^2$$

Either 2 disks with different radii if $g_{12} < \frac{g_1+\sqrt{g_1^2+8g_1g_2}}{4}$ (if $g_1 \leq g_2$), or a disk and an annulus. Convergence in the TF limit. Aftalion-Noris-Sourdis following Aftalion-Jerrard-Letelier and Karali-Sourdis:

• uniqueness of the ground state and of solutions with decay. We use the division trick of Lassoued-Mironescu to prove uniqueness. No moving plane method works to get radial symmetry.

• precise estimate of the convergence to the Thomas-Fermi limit. Proved by constructing an approximate solution. Then using the linearized operator, we perturb it to a genuine solution. By uniqueness, it is the ground state.

Inside the Thomas-Fermi radius: convergence in $\varepsilon^2|\log\varepsilon|$.

Outside: exponentially small.

Size of $\varepsilon^{1/3}$ around the Thomas-Fermi radius.

A full asymptotic expansion to any order has been proved by C.Gallo.

Case with rotation

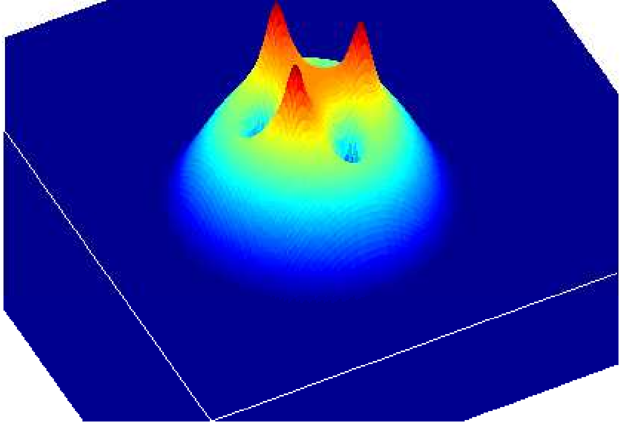
• until the first vortex, the minimizer is unique and real valued. Division of the ground state at Ω, by the ground state at $\Omega = 0$. Jacobian estimates provide that the ratio is 1. It means that the ground state stays real valued until the first vortex. (Aftalion-Noris-Sourdis).

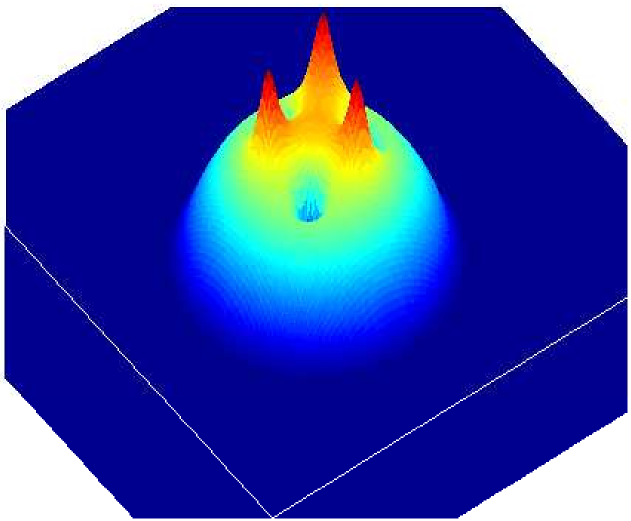
• vortex peak interaction. The equation of the vortex core has to be replaced by a system of vortex/spike $(f(r)e^{i\theta}, S(r))$ (Aftalion-Wei) Existence of a non degenerate solution

$$\frac{(rf')'}{r} - \frac{f}{r^2} + \alpha_1 f(1 - f^2) + \alpha_{12} S^2 f = 0,$$

$$\frac{(rS')'}{r} + \alpha_2 S(1 - S^2) + \alpha_{12} f^2 S = 0.$$

Related results by Eto, Kasamatsu, Nitta, Takeuchi, Tsubota, in the case of a homogeneous condensate and by Alama-Bronsard-Mironescu.

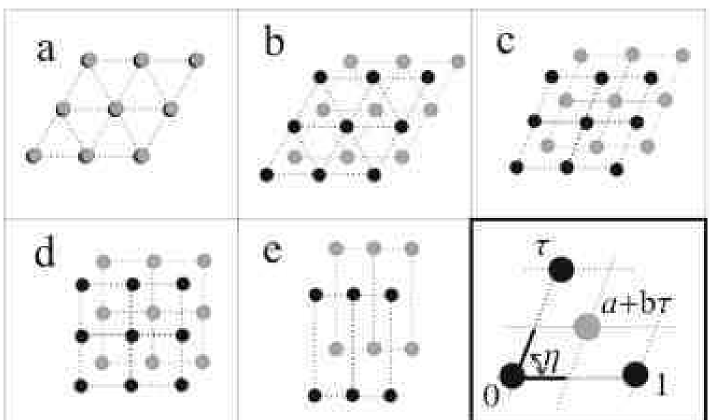




$$-\sum_{i,j}(\log|p_i-p_j|+\log|q_i-q_j|)+\sum_i(|p_i|^2+|q_i|^2)+\sum_{i,j}\frac{c_\Omega}{|p_i-q_j|^2}$$

where p_i are the vortices for component 1, q_j are the vortices for component 2 and $c_\Omega = \frac{\pi(1-\Gamma_{12})|\log g_1|^2}{8\Gamma_{12}^2 N_1 g_1}(2\frac{\Omega}{\Omega_c}-1)$. At some critical value of c_Ω, the lattice goes from triangular to square: relation between Γ_{12} and Ω.

From Kasamatsu-Tsubota-Ueda, vortex lattices analyzed formally using Theta functions. Work in progress with L.Betermin



Segregation case

If $\Gamma_{12} = 1 - \frac{g_{12}^2}{g_1 g_2} < 0$, phase separation is expected.

• $\Gamma_{12} \to -\infty$, $g_1 = g_2$: Gamma convergence to a De Giorgi type problem (Aftalion- Royo-Letelier). Slicing device of Ambrosio-Tortorelli

Limiting problem defined by the inverted parabola $\eta^2 = (\lambda - \frac{1}{2}r^2)_+$, where D is the disk of radius $\sqrt{\lambda}/2$ and $\int_D \eta^2 = N_1 + N_2$.

Find the optimal D_1 and D_2 such that

$D = D_1 \cup D_2$, $\int_{D_1} \eta^2 = N_1$, $\int_{D_2} \eta^2 = N_2$ and they minimize

$\int_{\partial D_1 \cap \partial D_2} \eta^3$.

Better to have half spaces than disk+annulus to minimize this interface integral, if N_1, N_2 are not too small.

In the case of no trapping, results about the limiting problem: Sternberg-Zumbrum and recently Alikakos-Faliagas.

Related results of Berestycki-Lin-Wei-Zhao (no trapping potential). See also for bounded domains Caffarelli-Lin, Dancer et al, Noris-Tavares-Terracini-Verzini.

• $\Gamma_{12} < 0$, finite, $g_1 = g_2$: Gamma convergence by Goldman, Royo-Letelier. $Min|\psi_1|^2 + |\psi_2|^2$ then the 2 goes down to m instead of 0.

• $\Gamma_{12} < 0$, finite, $g_1 \neq g_2$, then the limiting problem is on the Thomas Fermi profile: the two radii are not the same: in fact the disk+annulus is better (Goldman-Merlet) unless $|g_2 - g_1|$ is of order of ε.

Aftalion-Sourdis: analysis of the behaviour of the wave function near the interface. Λ large:

$$\begin{cases} -v_1'' + v_1^3 - v_1 + \Lambda v_2^2 v_1 = 0, \\ -v_2'' + v_2^3 - v_2 + \Lambda v_1^2 v_2 = 0, \end{cases} \tag{1}$$

$$(v_1, v_2) \to (0, 1) \text{ as } z \to -\infty, \quad (v_1, v_2) \to (1, 0) \text{ as } z \to +\infty. \tag{2}$$

The hyperbolic tangent is matched with the solution of the inner system to get an asymptotic expansion by using the properties of the linearized operator.

Spin orbit coupled condensates

$$\int \sum_{k=1,2} \left(\frac{1}{2}|\nabla \psi_k|^2 + \frac{1}{2} r^2 |\psi_k|^2 + \frac{g_k}{2}|\psi_k|^4 \right) + g_{12}|\psi_1|^2|\psi_2|^2$$

$$-\kappa \psi_1^* \left(i\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_2}{\partial y} \right) - \kappa \psi_2^* \left(i\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_1}{\partial y} \right)$$

under the constraint $\int |\psi_1|^2 + |\psi_2|^2 = 1$.

We assume $g_1 = g_2 = g$ and define $\delta = g_{12}/g$.

Aftalion-Mason, PRA 2013

We define $\rho_T = |\psi_1|^2 + |\psi_2|^2$, $\psi_k = \sqrt{\rho_T}\chi_k$, $\chi_k = |\chi_k|e^{i\theta_k}$ so that $|\chi_1|^2 + |\chi_2|^2 = 1$ and $S_z = |\chi_1|^2 - |\chi_2|^2$, $S_x = \chi_1^*\chi_2 + \chi_2^*\chi_1$, $S_y = -i(\chi_1^*\chi_2 - \chi_2^*\chi_1)$.

$\delta > 1$: segregation: at $\kappa = 0$, one component is empty. As κ increases, to a giant skyrmion (disk+ thin annulus circulation 1), to multiple annuli and eventually stripes.

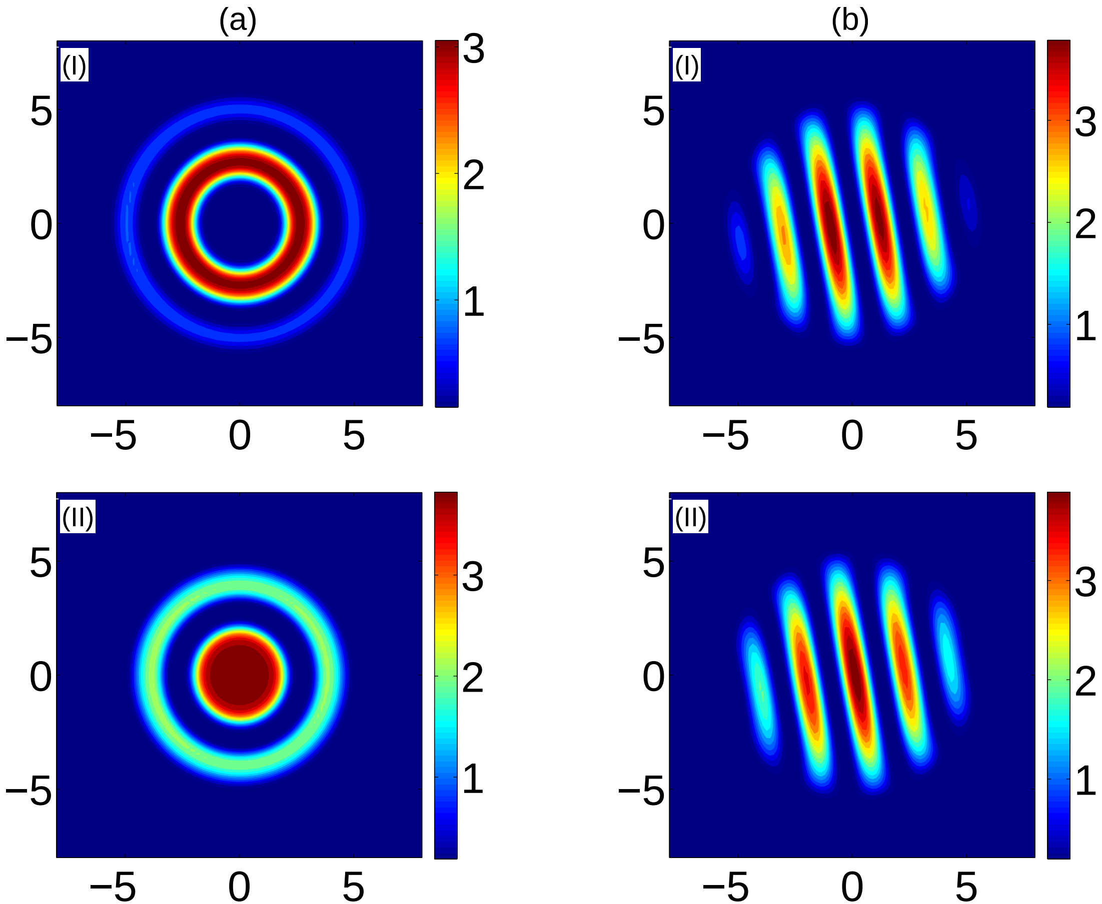


Figure 1: Left column (a): $(\delta, \kappa) = (1.5, 1.25)$ and right column (b): $(\delta, \kappa) = (1.5, 1.5)$. Density plots (frame (I), component-1, and (II), component-2).

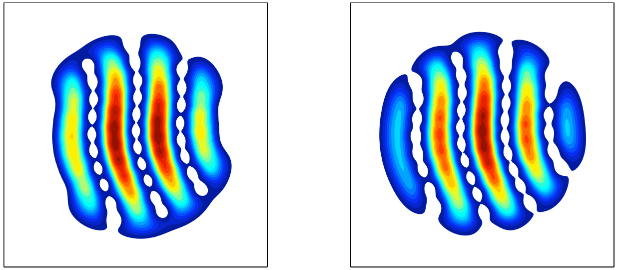
Question: understand the Gamma limit of the spin orbit term in the segregation case?

$$-\kappa\psi_2^*\left(i\frac{\partial\psi_1}{\partial x}-\frac{\partial\psi_1}{\partial y}\right)$$

and how it competes with the term $\int_{interface}\eta^3$.

Formally in the case disk+annulus, we find that the circulation in each annulus is 1.

Vortex sheets



Add rotation. This requires to balance the interface problem (minimal length) with the vortex contribution. Work in progress with E.Sandier, where we hope to prove that there is a regime of sheets (length can become large, and a rescaled rotation under the first critical field).

Happy birthday David!