ERC Readi meeting,
EHESS, Paris, December 2018

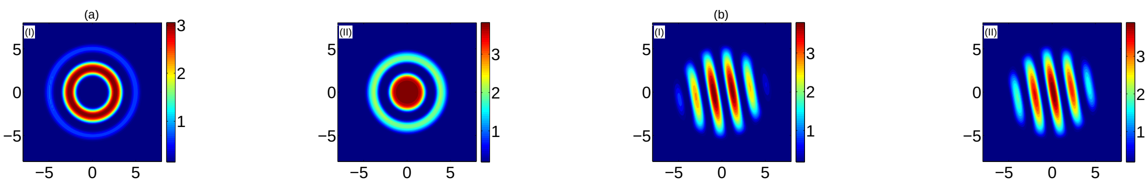
Segregation for coupled Bose Einstein condensates : recent results and open questions

Amandine Aftalion

CNRS, Centre d'Analyse et de Mathématique Sociales,
EHESS, Paris, France

Joint works with

Rémy Rodiac, Preprint 2017,
Etienne Sandier, Preprint 2018,
Christos Sourdis, Preprint 2018 and Com. Contemp. Math. 2017.



Motivated by numerical simulations in

Aftalion-Mason (PRA 2012 and 2013).

A two component Bose Einstein condensate is a mixture of 2 species:

- 2 different isotopes of the same alkali atom,

- or isotopes of different atoms,

- or a single isotope in 2 different hyperfine spin states.

Described by 2 wave functions ψ_1 and ψ_2 with $\int |\psi_1|^2 = N_1$, $\int |\psi_2|^2 = N_2$, or $\int |\psi_1|^2 + \int |\psi_2|^2 = N_1 + N_2$,
minimizing a Gross Pitaevskii energy with a coupling term.

The coupling is through the modulus and can be through the phase (spin orbit coupling or Rabi coupling).

Segregation or coexistence?

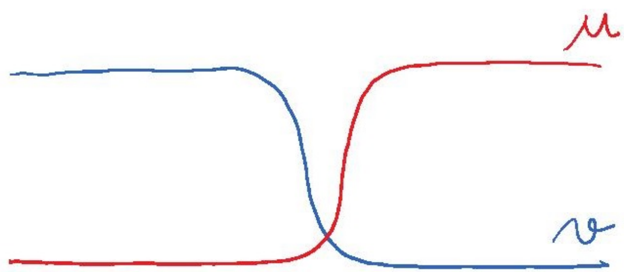
Segregation case: analysis of the behaviour of the wave function near the interface:

$$\begin{cases} -u'' + u^3 - u + \Lambda v^2 u = 0, \\ \\ -v'' + v^3 - v + \Lambda u^2 v = 0, \end{cases}$$

$$(u, v) \to (0, 1) \text{ as } z \to -\infty, \quad (u, v) \to (1, 0) \text{ as } z \to +\infty.$$

Λ large means strong segregation, $uv \to 0$.

Key analysis of the reduced system with coupling Berestycki-Lin-Wei-Zhao, Berestycki-Terracini-Wang-Wei



$$\begin{cases} -u'' + u^3 - u + \Lambda v^2 u = 0, \\ \\ -v'' + v^3 - v + \Lambda u^2 v = 0, \end{cases}$$

$$(u,v) \to (0,1) \text{ as } z \to -\infty, \quad (u,v) \to (1,0) \text{ as } z \to +\infty.$$

Aftalion-Sourdis, Λ large: the hyperbolic tangent (outer solution) is matched with the solution of the inner system to get an asymptotic expansion by using the properties of the linearized operator.

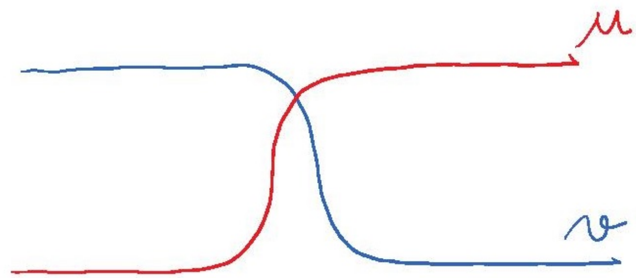
We prove that our solutions are linearly nondegenerate, and that there is a spectral gap, independent of Λ, between the zero eigenvalue (due to translations) at the bottom of the spectrum and the rest of the spectrum.

We prove a uniqueness result which implies that, in fact, the constructed heteroclinic is the unique minimizer (modulo translations) of the associated energy.

Farina-Sciunzi-Soave prove $u' < 0$, $v' > 0$ and uniqueness with the moving plane methods.

Sourdis, $\Lambda > 1$, $\Lambda \to 1$: slow-fast system analyzed with geometric singular perturbation theory.

We expect $u^2 + v^2 \to 1$ but have no bound for the convergence.



If $\Lambda \in (0, 1)$ (coexistence), Farina-Sciunzi-Soave prove that the solutions are equal and constants.

Rabi coupling

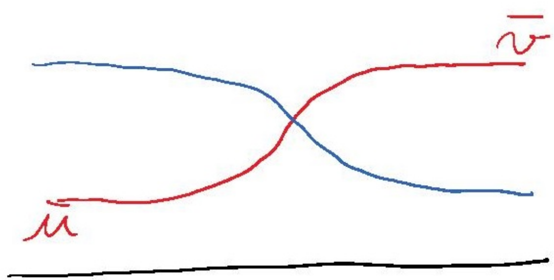
$$\begin{cases} -u'' + u^3 - u + \Lambda v^2 u = \omega v, \\ \\ -v'' + v^3 - v + \Lambda u^2 v = \omega u, \end{cases}$$

$$(u, v) \to (\bar{u}, \bar{v}) \text{ as } z \to -\infty, \quad (u, v) \to (\bar{v}, \bar{u}) \text{ as } z \to +\infty.$$

Competition between interaction and Rabi term.

ω acts against segregation so that large ω favors coexistence even when Λ is large!

We look for stable points...



$$\begin{cases} u(u^2+v^2-1)+v((\Lambda-1)uv-\omega)=0, \\ \\ v(u^2+v^2-1)+u((\Lambda-1)uv-\omega)=0, \end{cases}$$

If $c_0 = \frac{\omega}{\Lambda-1} \leq \frac{1}{2}$, two solutions $(\bar{u},\bar{v})$ and $(\bar{v},\bar{u})$ with $\bar{u}^2+\bar{v}^2=1$, $\bar{u}\bar{v}=c_0$. Phase transition is possible.

If $c_0 = \frac{\omega}{\Lambda-1} \geq \frac{1}{2}$, only $(0,0)$ and $(\sqrt{\frac{\omega+1}{\Lambda+1}}, \sqrt{\frac{\omega+1}{\Lambda+1}})$. Coexistence is expected.

Conjecture: if $c_0 = \frac{\omega}{\Lambda-1} \geq \frac{1}{2}$, then $u \equiv v$.

if $c_0 = \frac{\omega}{\Lambda-1} \leq \frac{1}{2}$, then there is a unique heteroclinic solution and it is monotone.

Difficulty: write the system for u^2+v^2-1 and $uv-c_0$ and get signs for these functions.

$$(uv)'' = (u^2+v^2)\left((\Lambda-1)uv-\omega\right) + 2uv\left(u^2+v^2-1\right) + 2u'v'$$

$$(u^2+v^2)'' = 2\left[(u^2+v^2)(u^2+v^2-1) + 2uv\left((\Lambda-1)uv-\omega\right)\right] + 2u'^2 + 2v'^2$$

We can prove (Aftalion-Sourdis) that there exists a solution with asymptotic properties.

If Λ is large and $\omega/(\Lambda-1)$ tends to $c_0 \in (0,1/2)$, then there exists a solution (u_Λ, v_Λ) such that $u_\Lambda v_\Lambda \to c_0$ and u_Λ tends to the solution of

$$(u')^2 + \frac{c_0^2 (u')^2}{u^4} = \frac{1}{2}(1 - u^2 - \frac{c_0^2}{u^2})^2$$

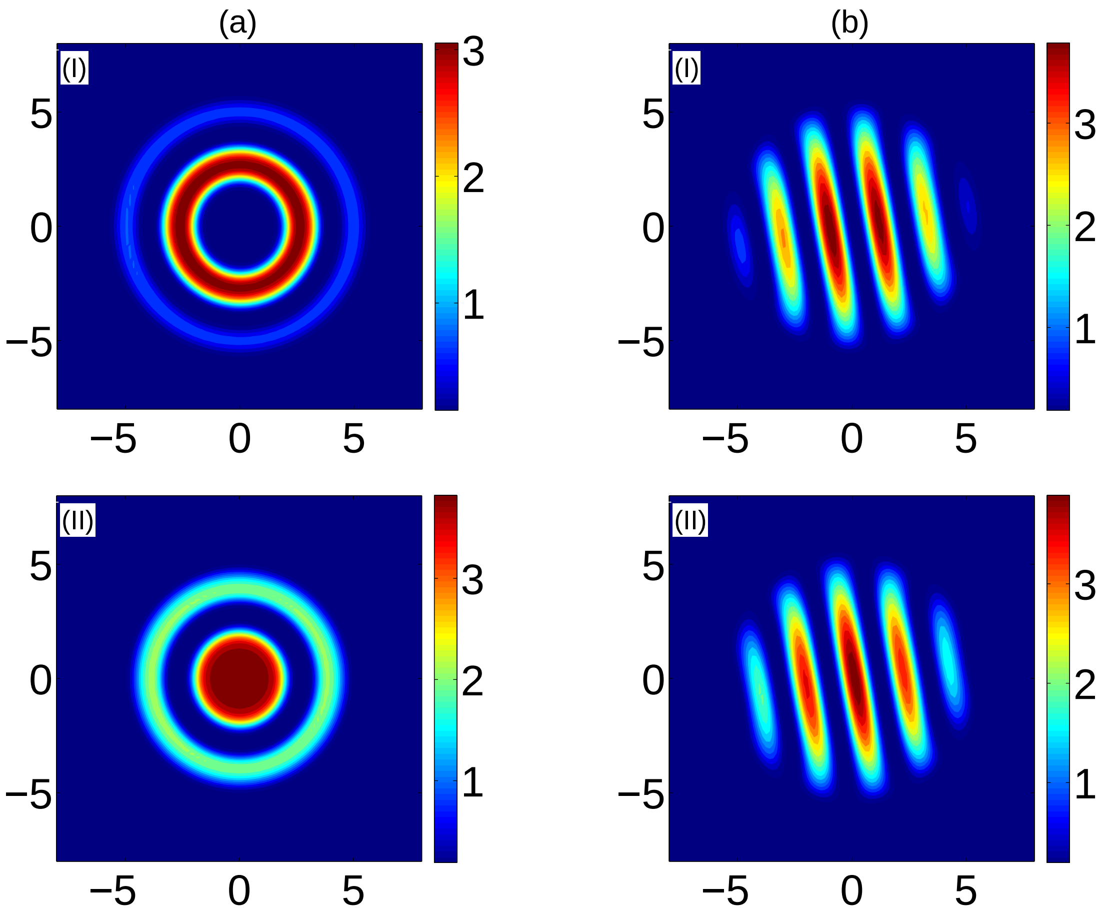
with $u(0) = \sqrt{c_0}$. In particular, $u(x)u(-x) = c_0$.

If $\Lambda - 1 \to 0^+$ and $\omega/(\Lambda-1)$ tends to $c_0 \in (0,1/2)$, then there exists a solution (u_Λ, v_Λ) such that $u_\Lambda^2 + v_\Lambda^2 \to 1$ and if $u_\Lambda = R_\Lambda \cos\phi_\Lambda/2$, $v_\Lambda = R_\Lambda \sin\phi_\Lambda/2$, then ϕ_Λ tends to the solution of

$$\phi' = (2c_0 - \sin\phi)$$

with $\phi(0) = \pi/2$. Moreover $\phi(x) + \phi(-x) = \pi$.

Spin orbit coupled condensates



S

Joint work with R.Rodiac. 1D analysis of stripes.

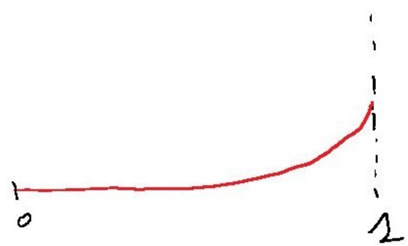
$$F_{\beta,\kappa}(\varphi) = \frac{1}{8}\int_0^1 {\varphi'}^2 + \frac{1}{\beta^2}\sin^2\varphi - \frac{\kappa}{2}\int_0^1 \varphi'$$

Modified Modica-Mortola type problem with $\varphi'(1) = 2\kappa$, $\varphi(0) = 0$.

Let $\tilde\kappa = \kappa\beta$. If $\tilde\kappa < 1/\pi$, then the minimizer is increasing and $\varphi(1-\beta x)$ converges to

$$2\arctan\left[\tan\left(\frac{\arcsin(2\tilde\kappa)}{2}\right)e^{-x}\right]$$

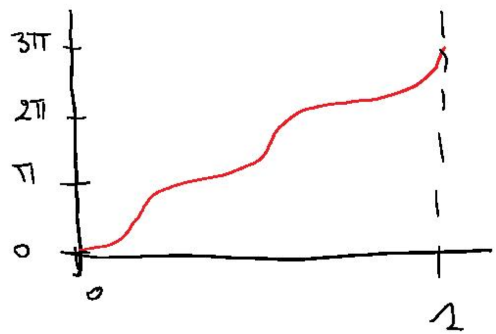
which stays below $\pi/2$.



If $\tilde{\kappa} > 1/\pi$, then the minimizer goes beyond π. In fact, there are N periods, with N of order $1/\beta$: stripes! $\varphi(\beta x)$ converges to the solution of

$$\left\{ \begin{array}{lll} \varphi_0'' & = & \sin\varphi_0 \cos\varphi_0 \text{ in } \mathbb{R}^+, \\ \varphi_0(0) & = & 0, \\ \varphi_0'(0) & = & \alpha_0, \end{array} \right. \tag{1}$$

where α_0 minimizes the energy per period.



If $\tilde{\kappa} < 1/\pi$, proof using Modica-Mortola:

$$F_{\beta,\kappa}(\varphi) \geq \frac{1}{4\beta} \int |\varphi'||\sin\varphi| - \frac{\tilde{\kappa}}{2\beta}\varphi(1)$$

Let $N = E(\varphi(1)/\pi)$.

$$F_{\beta,\kappa}(\varphi) \geq \frac{N}{2\beta}(1 - \tilde{\kappa}\pi) + \frac{1}{4\beta}(1 - \cos(\varphi(1) - N\pi) - 2\tilde{\kappa}(\varphi(1) - N\pi))$$

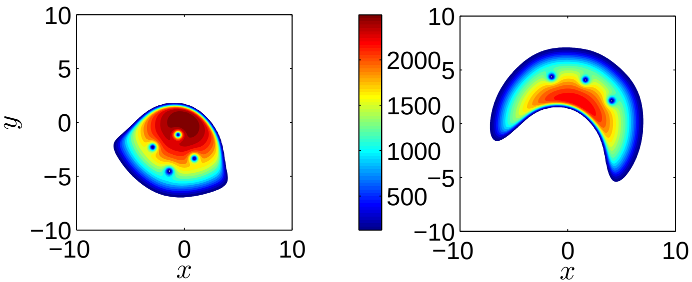
If $\tilde{\kappa} > 1/\pi$, proof using $\varphi' = \sqrt{\sin^2\varphi/\beta^2 + \varphi'(0)^2}$. This leads to the definition of α_0 given by

$$\tilde{\kappa}\pi = \int_0^{\pi/2} \sqrt{\alpha_0^2 + \sin^2 y}\; dy$$

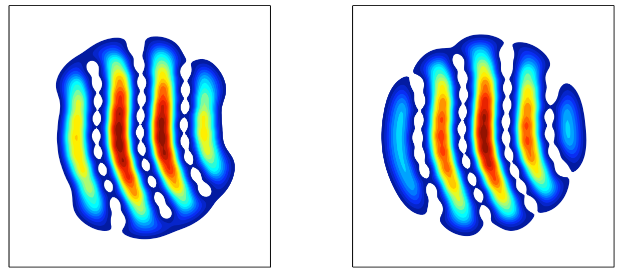
and minimizes the energy per period.

Segregation+rotation

Small rotation: minimizing interface with vortices in each droplet



Large rotation: Vortex sheets This requires to balance the interface problem (minimal length) with the vortex contribution. Work in progress with E.Sandier.



If $\Omega = 0$, let $(u_{1,\varepsilon}, u_{2,\varepsilon})$ be a minimizer of

$$F_\varepsilon(u_1,u_2) = \int_D \frac{1}{2}|\nabla u_1|^2 + \frac{1}{2}|\nabla u_2|^2 + W_\varepsilon(|u_1|,|u_2|),$$

$$W_\varepsilon(|u_1|,|u_2|) = \frac{1}{4\varepsilon^2}(1-|u_1|^2-|u_2|^2)^2 + \frac{\delta-1}{2\varepsilon^2}|u_1|^2|u_2|^2$$

under the constraint $\int_D |u_1|^2 = \alpha|D|$, $\int_D |u_2|^2 = (1-\alpha)|D|$.

Note that $\delta \to 1$ and $\tilde{\varepsilon} = \frac{\varepsilon}{\sqrt{\delta-1}} << \varepsilon$.

The limit is given by minimizing the perimeter of the interface with the mass constraint, and $(|u_{1,\varepsilon}|, |u_{2,\varepsilon}|)$ converge to $(\chi_{\omega_\alpha}, \chi_{\omega_\alpha^c})$, where ω_α minimizes the perimeter with volume α.

We want to estimate the error term in the Modica Mortola energy

Difficulty: bound for the scalar energy outside a neighborhood of the interface. Related results by Leoni-Murray, for the error term in the Γ convergence for the Modica Mortola problem.

Lemma: If

$$F_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}) \leq \frac{l(\omega)}{\tilde{\varepsilon}} + \Delta_\varepsilon$$

where $\Delta_\varepsilon << \frac{l(\omega)}{\tilde{\varepsilon}}$, then $(u_{1,\varepsilon}, u_{2,\varepsilon})$ converge to $(\chi_{\omega_\alpha}, \chi_{\omega_\alpha^c})$ and for any η, there exists C such that for ε small enough, there is V_η an η neighborhood of the interface such that

$$F_\varepsilon(u_{1,\varepsilon}, u_{2,\varepsilon}, D \setminus V_\eta) \leq C(\Delta_\varepsilon + |\log \tilde{\varepsilon}|).$$

The potential W_ε has two minima $a = (1,0)$ and $b = (0,1)$. We define the distance to a as the energy to connect to a:

$$d_\varepsilon(x,a) = \inf_\gamma \left\{ \int_{-\infty}^0 |\gamma'(t)|^2 + W(\gamma(t))\, dt \lim_{-\infty} \gamma = a,\ \ \gamma(0) = x \right\}.$$

Then from the coarea formula

$$F_\varepsilon(u) \geq \int_0^{d_\varepsilon(a,b)} per_D \{ d_\varepsilon(u(x)) < t \}\ \ dt.$$

If $\Omega \sim \beta|\log\tilde{\varepsilon}|$, then

$$\min E_{\varepsilon}^{\Omega} \sim \frac{l(\omega)}{\tilde{\varepsilon}} + \Omega^2 C(\omega)(1 + o(1))$$

If $|\log\tilde{\varepsilon}| << \Omega$ and $\Omega\log(\frac{1}{\tilde{\varepsilon}\Omega}) << \frac{1}{\tilde{\varepsilon}}$, then

$$\min E_{\varepsilon}^{\Omega} \sim \frac{l(\omega)}{\tilde{\varepsilon}} + \frac{1}{2}|D|\Omega\log(\frac{1}{\tilde{\varepsilon}\Omega})(1 + o(1))$$

Uniform vortex distribution in each component

For larger Ω, the vortex contribution is the leading order term.

Open questions

Contribution of the interface term when it is not leading order, Vortex serpentine

Dissipation in superfluid mixtures: polaritons, project with physicists from LKB, ENS (F.Chevy and S.Pigeon).

superfluid flow

$$\mathrm{div}\left(n(|\nabla\theta|)\,\nabla\theta\right) = 0$$

helium $n(|\nabla\theta|) = n_0 - |\nabla\theta|^2$

polariton

$$n\sqrt{n} + \frac{|\nabla\theta|^2 - \Delta}{g}\sqrt{n} - \frac{F\cos(\theta + kx)}{g} = 0$$